

MITSCH'S ORDER AND INCLUSION FOR BINARY RELATIONS AND PARTITIONS

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ABSTRACT. Mitsch's natural partial order on the semigroup of binary relations is here characterised by equations in the theory of relation algebras. The natural partial order has a complex relationship with the compatible partial order of inclusion, which is explored by means of a sublattice of the lattice of preorders on the semigroup. The corresponding sublattice for the partition monoid is also described.

1. NATURAL PARTIAL ORDERS

In [10], Heinz Mitsch formulated a characterisation of the natural partial order \leq on the full transformation semigroup \mathcal{T}_X which did not use inverses or idempotents, and went on to define the natural partial order \leq on any semigroup S by

$$(1) \quad a \leq b \text{ if } a = b \text{ or there are } x, y \in S \text{ such that } a = ax = bx = yb$$

for $a, b \in S$. (The discovery was also made independently by P. M. Higgins, but remained unpublished.) Observe that $a = ya$ follows. Mitsch's natural partial order has now been characterised, and its properties investigated, for several concrete classes of non-regular semigroups—in [8, 12] for some semigroups of (partial) transformations, and by Namnak and Preechasilp [11] for the semigroup \mathcal{B}_X of all binary relations on the set X .

The partial order of inclusion which is carried by \mathcal{B}_X may also be thought of as 'natural', and it is the broad purpose of this note to discuss the relationship between these two partial orders on \mathcal{B}_X . Moreover, the same questions are addressed for the partition monoid \mathcal{P}_X , which also carries two 'natural' partial orders. So we shall use a slightly different nomenclature here for the sake of clarity, mostly referring to partial orders as just *orders*, and the natural partial order as *Mitsch's order*. We begin by collecting some information about \mathcal{B}_X .

2. BINARY RELATIONS

The notation used here for binary relations follows that found in, for example, Clifford and Preston [1], with the addition of complementation of relations defined by

$$x \alpha^c y \iff (x, y) \notin \alpha$$

for $x, y \in X$. Note that the symbol \circ for composition will be suppressed, except for the composites of order relations on \mathcal{B}_X . We will make use of the identity relation on X , $\iota = \{(x, x) : x \in X\}$, and the universal relation $\omega = X \times X$.

The following logical equivalence will also be required; it is the 'Theorem K' of De Morgan [2, p. xxx] (see also e.g. [6]).

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Result 2.1. For $\alpha, \beta, \xi \in \mathcal{B}_X$,

$$\beta\xi \subseteq \alpha \iff \xi \subseteq (\beta^{-1}\alpha^c)^c \iff \beta \subseteq (\alpha^c\xi^{-1})^c.$$

Result 2.1 will be used in the following form:

Corollary 2.2. (i) If the set $\{\xi \in \mathcal{B}_X : \beta\xi = \alpha\}$ is non-empty, it has greatest element $(\beta^{-1}\alpha^c)^c$ in the inclusion order.
(ii) If the set $\{\xi \in \mathcal{B}_X : \xi\beta = \alpha\}$ is non-empty, it has greatest element $(\alpha^c\beta^{-1})^c$ in the inclusion order.

Proof. (i) By Result 2.1 above, $\beta\xi = \alpha$ implies $\xi \subseteq (\beta^{-1}\alpha^c)^c$; but the latter implies, again using Result 2.1, that

$$\alpha = \beta\xi \subseteq \beta(\beta^{-1}\alpha^c)^c \subseteq \alpha.$$

Part (ii) is proven dually. □

3. EQUATIONAL CRITERION FOR THE MITSCH ORDER ON \mathcal{B}_X

The basic definition (1) of $\alpha \leq \beta$ is existentially quantified. In \mathcal{B}_X there is a purely equational equivalent:

Theorem 3.1. For $\alpha, \beta \in \mathcal{B}_X$, $\alpha \leq \beta$ if and only if

$$\alpha = \alpha(\beta^{-1}\alpha^c)^c = \beta(\beta^{-1}\alpha^c)^c = (\alpha^c\beta^{-1})^c\beta.$$

Proof. Suppose $\alpha \leq \beta$. By definition there are ξ, η such that

$$\alpha = \alpha\xi = \beta\xi = \eta\beta$$

and hence both

$$\eta\alpha = \eta\beta\xi = \alpha\xi = \alpha$$

and

$$\alpha = \beta(\beta^{-1}\alpha^c)^c = (\alpha^c\beta^{-1})^c\beta,$$

the latter by Corollary 2.2. Now

$$\begin{aligned} \alpha &= \eta\alpha = \eta\beta(\beta^{-1}\alpha^c)^c = \alpha(\beta^{-1}\alpha^c)^c \\ &= \beta(\beta^{-1}\alpha^c)^c = (\alpha^c\beta^{-1})^c\beta. \end{aligned}$$

Conversely,

$$\alpha = \alpha(\beta^{-1}\alpha^c)^c = \beta(\beta^{-1}\alpha^c)^c = (\alpha^c\beta^{-1})^c\beta$$

demonstrates $\alpha \leq \beta$. □

Regarded as a computation, this criterion is of polynomial time complexity in $|X|$, as is also the case for tests of the divisibility preorders (by Corollary 2.2, $\alpha = \beta\xi$ if and only if $\alpha = \beta(\beta^{-1}\alpha^c)^c$, etc.), but in contrast to the NP-complete tests for the \mathcal{J} -preorder [7]. Of course algorithmic complexity is not the only issue. Namnak and Preechasilp [11] characterise Mitsch's order for binary relations with the aid of Zaretskiĭ's criteria for divisibility [14] which, though also of worst-case exponential complexity, proved very convenient for the purposes of finding compatible elements, atoms and maximal elements in the Mitsch order [11]. The equations of Theorem 3.1 are complex in the different sense that they belong to a theory of semigroups enriched by operations of inversion and complementation, in fact, to the theory of relation algebras [6]. It may be observed that all the

proofs of sections 4 and 5 apply to the multiplicative reducts of (abstract) relation algebras [6], and not just the representable ones \mathcal{B}_X .

4. CONNEXIONS WITH THE INCLUSION ORDER

In discussing Mitsch's order and the inclusion order on \mathcal{B}_X , [11] notes their logical independence. In fact, we can see that every inclusion atom is a Mitsch atom, and every non-empty relation on X is in an inclusion interval between two Mitsch atoms. Moreover, permutations of X are Mitsch-maximal, but far from being either maximal elements or atoms in the inclusion order. Yet the relationship between the two orders is subtle, and worthy of further exploration. We illustrate this by next finding a substructure of \mathcal{B}_X in which \leq agrees with \subseteq , and others where \leq agrees with reverse inclusion \supseteq . We shall use the following statement, easily proved.

Lemma 4.1. *Let S be a semigroup, T be a regular subsemigroup of S , and $a, b \in T$. Then $a \leq b$ in T if and only if $a \leq b$ in S .*

The symmetric inverse monoid \mathcal{I}_X is a regular subsemigroup of \mathcal{B}_X , and the natural partial order on \mathcal{I}_X coincides with inclusion. So we may apply Lemma 4.1 to \mathcal{I}_X .

Corollary 4.2. *(i) If $\alpha, \beta \in \mathcal{I}_X$, then $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.
(ii) If $\beta \in \mathcal{I}_X$, then $\alpha \subseteq \beta$ implies $\alpha \leq \beta$.*

Part (ii) relies on the observation that $\beta \in \mathcal{I}_X$ and $\alpha \subseteq \beta$ imply $\alpha \in \mathcal{I}_X$. Of course, $\alpha \leq \beta$ need not entail $\alpha \in \mathcal{I}_X$.

To obtain pairs (α, β) such that \leq agrees with reverse inclusion \supseteq , first let α be a reflexive and transitive relation on X (so, a preorder), and define a subset of \mathcal{B}_X by

$$F(\alpha) = \{\beta \in \mathcal{B}_X : \alpha\beta = \alpha = \beta\alpha\}.$$

Proposition 4.3. *For all $\beta \in F(\alpha)$, $\alpha \leq \beta$ if and only if $\beta \subseteq \alpha$.*

Proof. Clearly $F(\alpha)$ is a subsemigroup of \mathcal{B}_X , with zero element α . So for all $\beta \in F(\alpha)$, we have $\alpha \leq \beta$ and, since $\iota \subseteq \alpha$, also $\beta \subseteq \beta\alpha = \alpha$. \square

Corollary 4.2 shows that it may be instructive to consider the conjunction of the natural partial order with set inclusion, which is an order on \mathcal{B}_X which we naturally write as $\leq \cap \subseteq$. Similarly, Proposition 4.3 suggests investigating the conjunction of \leq with reverse inclusion, an order written as $\leq \cap \supseteq$. The next Proposition, which extends Proposition 4.3, gives an alternative characterisation for $\leq \cap \supseteq$; there seems to be no analogous description of $\leq \cap \subseteq$.

Proposition 4.4. *For all $\alpha, \beta \in \mathcal{B}_X$, $\alpha \leq \beta$ and $\alpha \supseteq \beta$ if and only if there are $\varepsilon = \varepsilon^2$ and $\phi = \phi^2$ such that $\iota \subseteq \varepsilon$ and $\alpha = \beta\varepsilon = \phi\beta$.*

Proof. Let $\alpha \leq \beta$. From Corollary 2.2 there exists the element ε , maximum with respect to \subseteq such that $\alpha = \beta\varepsilon$, and $\alpha = \alpha\varepsilon$ also holds by Theorem 3.1. Then also $\alpha = \beta\varepsilon^2$ and thus $\varepsilon^2 \subseteq \varepsilon$. But from $\beta\iota \subseteq \alpha$ we have $\iota \subseteq \varepsilon$, so $\varepsilon \subseteq \varepsilon^2$ and $\varepsilon = \varepsilon^2$. Similarly $\phi = \phi^2$ with $\alpha = \phi\beta$.

Conversely, if the conditions hold then plainly $\alpha \leq \beta$ as in the regular case, but also $\iota \subseteq \varepsilon$ implies $\beta \subseteq \varepsilon\beta = \alpha$. \square

5. THE SUBLATTICE OF PREORDERS GENERATED BY \leq , \subseteq AND \supseteq

The subsemigroups $F(\alpha)$ and \mathcal{I}_X of section 4 clearly show that, if $|X| \geq 2$, there can be no order relation on \mathcal{B}_X which contains \leq and either of \subseteq and \supseteq . However the set of preorders on \mathcal{B}_X is bounded by the relation of equality $=$ and the universal relation on \mathcal{B}_X , and is closed under arbitrary intersections, and so forms a lattice. In a similar situation, the papers [8] and [12] derive interesting results from considering the composite of inclusion with \leq , and this idea also turns out to be useful here.

Proposition 5.1. *For all $\alpha, \beta \in \mathcal{B}_X$, there exists $\gamma \in \mathcal{B}_X$ such that $\alpha \subseteq \gamma \leq \beta$ if and only if $\alpha\omega\alpha \subseteq \beta\omega\beta$.*

Proof. Suppose $\alpha \subseteq \gamma \leq \beta$. Then there are ξ, η such that $\gamma = \beta\xi = \eta\beta$, so

$$\alpha\omega\alpha \subseteq \gamma\omega\gamma = \beta\xi\omega\eta\beta \subseteq \beta\omega\beta.$$

Since $\beta \subseteq \beta\beta^{-1}\beta \subseteq \beta\omega\beta$, we have

$$\beta\omega\beta \subseteq \beta\omega\beta\omega\beta \subseteq \beta\omega\beta.$$

But now $\beta\omega\beta = \beta\omega\beta\omega\beta$ shows that $\beta\omega\beta \leq \beta$. As above we have $\alpha \subseteq \alpha\omega\alpha$, so $\alpha\omega\alpha \subseteq \beta\omega\beta$ implies $\alpha \subseteq \beta\omega\beta \leq \beta$. \square

As a corollary, we have the join of the Mitsch and inclusion orders in the lattice of preorders on \mathcal{B}_X .

Corollary 5.2. (i) *The composite $\leq \circ \subseteq$ is contained in $\subseteq \circ \leq$.*
(ii) *$\subseteq \circ \leq$ is the join of \subseteq and \leq in the lattice of preorders on \mathcal{B}_X .*

Proof. (i) First let us note that \leq and \subseteq are subsets of $\subseteq \circ \leq$. It is clear from Proposition 5.1 that $\subseteq \circ \leq$ is transitive, i.e.,

$$(\subseteq \circ \leq) \circ (\subseteq \circ \leq) = (\subseteq \circ \leq)$$

and it follows that $(\leq \circ \subseteq)$ is contained in $(\subseteq \circ \leq)$.

(ii) Also it is immediate that $\subseteq \circ \leq$ is reflexive, and so it is a preorder on \mathcal{B}_X . Any preorder on \mathcal{B}_X containing both \subseteq and \leq also contains $\subseteq \circ \leq$. Hence $\subseteq \circ \leq$ is the join of \subseteq and \leq . \square

That the containment in (i) is proper (for $|X| \geq 2$) is shown by the following instance. Consider a pair of distinct permutations π, ρ : we have $\pi \subseteq \omega \leq \rho$, but $\pi \leq \alpha \subseteq \rho$ implies $\pi = \alpha = \rho$, a contradiction. It also follows that $\leq \circ \subseteq$ is not a preorder.

We turn to the composites and join of \leq with reverse inclusion.

Proposition 5.3. *The composite $\supseteq \circ \leq$ is the universal relation on \mathcal{B}_X and the join of \supseteq and \leq in the lattice of preorders on \mathcal{B}_X .*

Proof. For any $\alpha, \beta \in \mathcal{B}_X$, $\alpha \supseteq \emptyset \leq \beta$, so $\mathcal{B}_X \times \mathcal{B}_X$ coincides with $\supseteq \circ \leq$. Now $\mathcal{B}_X \times \mathcal{B}_X$ is plainly a preorder, and any preorder containing both \leq and \supseteq must contain $\supseteq \circ \leq$ and hence $\mathcal{B}_X \times \mathcal{B}_X$. \square

Here too, we see that the reverse composite $\leq \circ \supseteq$ is properly contained in $\supseteq \circ \leq = \mathcal{B}_X \times \mathcal{B}_X$ when $|X| \geq 2$ (and so is not a preorder), since $\iota \leq \alpha \supseteq \omega$ implies both $\alpha = \iota$ and $\alpha = \omega$.

Thus we are able to describe the relationships between \leq , \subseteq and \supseteq in terms of the sublattice generated by \leq , \subseteq and \supseteq within the lattice of preorders on \mathcal{B}_X . This sublattice is summarised by a Hasse diagram in Fig. 1; filled circles denote orders, and additional labels in parentheses summarise conditions for α, β to be related by the preorder.

Fig. 1. A sublattice of preorders on \mathcal{B}_X . Filled circles denote orders.

6. DUALISING: PARTITION MONOIDS

The partition monoid on the set X , denoted by \mathcal{P}_X , generates the partition algebra, which is important in group representation theory and statistical mechanics; for expositions, see [9] and [5]. Study of \mathcal{P}_X *qua* semigroup is more recent and scanty. The reader is referred to the articles [13], [3], [4], which include full descriptions and examples; here we give just a concise description, which suffices for the present purpose. A *partition over* X is a quotient object (or equivalently, a partition) of the coproduct (disjoint union) $X \sqcup X$ of two copies of the set X . It is often convenient to represent $a \in \mathcal{P}_X$ as a graph on the vertex set $X_0 \cup X_1$ (where X_0 and X_1 are disjoint copies of X) in which, for $i, j \in X_0 \cup X_1$, $\{i, j\}$ is an edge if and only if i, j belong to the same block of the partition a . (The resulting graph is then a union of cliques.) If a block of $a \in \mathcal{P}_X$ has elements in both X_0 and X_1 , we say it is a *transversal* block.

For the product in \mathcal{P}_X , let b be represented as a union of cliques on $X_1 \cup X_2$ (where X_2 is another copy of X disjoint from both X_0 and X_1). Thus we have a graph Γ on the vertex set $X_0 \cup X_1 \cup X_2$; and then we construct a graph ab on the subset of vertices $X_0 \cup X_2$ by declaring, for $i, j \in X_0 \cup X_2$, that $\{i, j\}$ is an edge of ab if and only if there is a path in Γ from i to j . Note ab is a union of cliques, and each path may be taken, without loss of generality, to have edges taken alternately from a and b , if null initial and terminal edges are allowed.

The relevance of \mathcal{P}_X in the present paper lies in its relationship with the semigroup of binary relations. It takes just a short paragraph to explain how the partition monoid \mathcal{P}_X is conceptually dual to the semigroup of binary relations; we begin with a categorical description of \mathcal{B}_X . A *relation from* X *to* Y over a category with products is a subobject of the product $X \times Y$. If the base category

also has pullbacks and images, there is a natural way of defining the product of a relation from X to Y with a relation from Y to Z . Over the category **Set** this product is the usual composition of relations, and so is associative; thus we derive the category of binary relations over **Set**, and the endomorphism monoid at the object X is simply \mathcal{B}_X . We could call \mathcal{B}_X the *relation monoid at X* over **Set**. Now the category $\mathbf{Set}^{\text{opp}}$ also turns out (non-trivially) to satisfy the conditions above. A product in $\mathbf{Set}^{\text{opp}}$ is the same as a coproduct of sets, and a subobject in $\mathbf{Set}^{\text{opp}}$ is a quotient object in **Set**. So elements of the relation monoid at X over $\mathbf{Set}^{\text{opp}}$ do indeed constitute the partition monoid \mathcal{P}_X as defined above.

More significantly, the multiplication operations in \mathcal{P}_X and \mathcal{B}_X are categorical dual constructs as well. The monoids \mathcal{B}_X and \mathcal{P}_X share the properties of containing, in a canonical way, the semigroups \mathcal{T}_X of all maps $X \rightarrow X$, and \mathcal{I}_X of all partial injective maps $X \rightarrow X$; \mathcal{P}_X additionally has the property of being a regular $*$ -semigroup, and containing, in a canonical way, the dual symmetric inverse monoid \mathcal{I}_X^* . Importantly for present purposes, \mathcal{P}_X also carries, like \mathcal{B}_X , an inclusion order, in which we write (for $a, b \in \mathcal{P}_X$) $a \subseteq b$ if and only if every edge of a is an edge of b . This order is the usual refinement order on set partitions, and is compatible with the product in \mathcal{P}_X :

Lemma 6.1. *Let $a, b, c \in \mathcal{P}_X$ with $a \subseteq b$. Then $ac \subseteq bc$ and dually $ca \subseteq cb$.*

Proof. Let i, \dots, j be a path in ac ; each edge from a is an edge of b , by hypothesis. So i, \dots, j is a path in bc . \square

In the canonical copy of \mathcal{I}_X in \mathcal{P}_X , the natural partial order agrees with inclusion, and in the canonical copy of \mathcal{I}_X^* , the natural partial order agrees with reverse inclusion. It follows, as with \mathcal{B}_X , that \leq and \subseteq are related in a complex way, which we may seek to understand in the same way as was used above for \mathcal{B}_X . In particular, we see again that $\leq \cap \subseteq$ and $\leq \cap \supseteq$ are non-trivial on \mathcal{P}_X but have a trivial conjunction; and likewise that there can be no partial order on \mathcal{P}_X which contains both \leq and \subseteq , or both \leq and \supseteq .

Just as in section 5, composites help identify joins of the orders within the lattice of preorders on \mathcal{P}_X . We need the following definitions: let \mathbf{d} be the trivial partition over X , thus corresponding to the discrete graph on $X \sqcup X$, and let \mathbf{k} be the universal partition over X , thus corresponding to the complete graph on $X \sqcup X$. Clearly, \mathbf{d} is the least element, and \mathbf{k} the greatest, in the inclusion order of \mathcal{P}_X .

Lemma 6.2. *For all $a \in \mathcal{P}_X$,*

- (i) $\mathbf{d}a\mathbf{d} = \mathbf{d}$;
- (ii) *if a has a transversal block, then $\mathbf{k}a\mathbf{k} = \mathbf{k}$;*
- (iii) *if a has no transversal block, then $\mathbf{k}a = a$.*

Proof. By direct computation. \square

Corollary 6.3. *For all $a \in \mathcal{P}_X$, $a\mathbf{d}$, $\mathbf{d}a$, $a\mathbf{k}$, $\mathbf{k}a$ are idempotents.*

Proposition 6.4. *For all $a, b \in \mathcal{P}_X$, $a \supseteq \circ \leq b$ if and only if $a\mathbf{d}a \supseteq b\mathbf{d}b$.*

Proof. Suppose there is $c \in \mathcal{P}_X$ such that $a \supseteq c \leq b$. Then $c = bx = yb$ for some $x, y \in \mathcal{P}_X$, and we have $a\mathbf{d}a \supseteq c\mathbf{d}c = by\mathbf{d}xb \supseteq b\mathbf{d}^3b = b\mathbf{d}b$.

For the converse, we see that $a\mathbf{d}a \supseteq b\mathbf{d}b$ implies $a = aa^*a \supseteq a\mathbf{d}a \supseteq b\mathbf{d}b \leq b$, the latter following from idempotency of $b\mathbf{d}$ and $\mathbf{d}b$. \square

Corollary 6.5. *In \mathcal{P}_X ,*

- (i) $\supseteq \circ \leq$ *is a preorder;*
- (ii) $\leq \circ \supseteq$ *is contained in $\supseteq \circ \leq$;*
- (iii) $\supseteq \circ \leq$ *is the join of \supseteq and \leq in the lattice of preorders.*

Proposition 6.6. *For all $a, b \in \mathcal{P}_X$, $a \subseteq \circ \leq b$ if and only if $aka \subseteq bkb$.*

Proof. Suppose there is $c \in \mathcal{P}_X$ such that $a \subseteq c \leq b$. Then there are $x, y \in \mathcal{P}_X$ such that $c = bx = yb$ and hence $aka \subseteq ckc = bykxb \subseteq bk^3b = bkb$.

Conversely, the condition implies $a = aa^*a \subseteq aka \subseteq bkb \leq b$, the latter following from idempotency of $b\mathbf{k}$ and $\mathbf{k}b$. \square

Corollary 6.7. *In \mathcal{P}_X ,*

- (i) $\subseteq \circ \leq$ *is a preorder;*
- (ii) $\leq \circ \subseteq$ *is contained in $\subseteq \circ \leq$;*
- (iii) $\subseteq \circ \leq$ *is the join of \subseteq and \leq in the lattice of preorders.*

As in the previous section, this information is summarised in Fig.2, a Hasse diagram depicting the sublattice of the lattice of preorders on \mathcal{P}_X generated by \leq, \subseteq , and \supseteq . Filled circles denote orders, and parentheses contain conditions for a and b to be related in the corresponding preorder.

Fig. 2. A sublattice of preorders on \mathcal{P}_X . Filled circles denote orders.

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